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Robert Rubinstein
NASA Langley Research Center, Hampton, Virginia

Ye Zhou
Lawrence Livermore National Laboratory, Livermore, California

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SCHIESTEL'S DERIVATION OF THE EPSILON EQUATION AND TWO EQUATION MODELING OF ROTATING TURBULENCE

ROBERT RUBINSTEIN* AND YE ZHOU†

Abstract. As part of a more general program of developing multiple-scale models of turbulence, Schiestel suggested a derivation of the homogeneous part of the dissipation rate transport equation. Schiestel's approach is generalized to rotating turbulence. The resulting model reproduces the main features observed in decaying rotating turbulence.

Key words. dissipation rate equation, rotating turbulence, two-equation turbulence models, multiple-scale turbulence models

Subject classification. Fluid Mechanics

1. Introduction. The dissipation rate transport equation continues to resist systematic derivation, either from the governing equations or even from statistical closures. Much of the closure-based work is summarized in [1]; more recent work is summarized in [2]. In many respects, the most successful derivation of the ϵ transport equation is due to Schiestel [3]. Among the successes of the derivation is a rather good value $C_{\epsilon 1} = 1.5$ and the demonstration that necessarily, $C_{\epsilon 2} > C_{\epsilon 1}$.

It is well-known that the derivation of the ϵ equation in rotating turbulence encounters additional difficulties because rotation does not appear explicitly in the exact transport equation for the dissipation rate. Instead, the effect of rotation is indirect, entering only through quantities like the turbulent time-scale. In the present work, the ϵ transport equation is treated by combining Schiestel's arguments with the phenomenology for rotating turbulence of Zhou [4]. The most direct generalization of the argument of [3] leads to a rotation-sensitized ϵ equation with the same form as the standard ϵ equation, but with an increased value of $C_{\epsilon 2}$; a model of this type was proposed by Okamoto [5]. A simple modification of the argument of [3] yields instead a model of the form first proposed by Bardina *et al* [6]. The implications of these models for decaying rotating turbulence are discussed.

2. Review of Schiestel's derivation. We begin with a split-spectrum model of high Reynolds number turbulence,

$$E(\kappa) = \begin{cases} C\kappa^2 & \text{if } \kappa < \kappa_0 \\ C_K \epsilon^{2/3} \kappa^{-5/3} & \text{if } \kappa > \kappa_0 \end{cases} \tag{2.1}$$

In Eq. (2.1), κ_0 is the inverse integral scale of turbulence which marks the transition between the inertial range and the large scales. Eq. (2.1) is a special case of the models introduced in [3] in connection with multiple-scale turbulence models. This is no more than a schematic model of the actual energy spectrum; however, as stressed in [3] and [7], to derive a two-equation model, it is essential that the spectrum be

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parametrized in some simple way. Use of a more complex model like the von Kàrmàn spectrum would lead to essentially the same results.

Denote the energy in the inertial range by

$$k = \frac{3}{2} C_K \epsilon^{2/3} \kappa_0^{-2/3} \tag{2.2}$$

and the energy in the large scales by

$$k_0 = \frac{1}{3} C \kappa_0^3 \tag{2.3}$$

Assume that the spectral descriptors in Eq. (2.1) are functions of time: $\epsilon = \epsilon(t)$ and $\kappa_0 = \kappa_0(t)$. It follows from Eq. (2.2) that

$$\dot{k} = C_K (\epsilon^{-1/3} \kappa_0^{-2/3} \dot{\epsilon} - \epsilon^{2/3} \kappa_0^{-5/3} \dot{\kappa}_0) \tag{2.4}$$

This equation does not lead to the desired ϵ equation directly, because it contains the new unknown $\dot{\kappa}_0$.

To solve this problem, we postulate

$$\dot{\kappa}_0 = -\beta \frac{\epsilon}{E(\kappa_0)} \tag{2.5}$$

based on a very similar proposal in [3]. In view of Eq. (2.1), Eq. (2.5) is equivalent to

$$\dot{\kappa}_0 / \kappa_0 = -\frac{3}{2} \beta \epsilon / k \tag{2.6}$$

Then Eqs. (2.4)–(2.6) give

$$\dot{k} = \frac{2}{3} \frac{k}{\epsilon} \dot{\epsilon} - \beta \epsilon \tag{2.7}$$

which can be re-arranged as

$$\dot{\epsilon} = \frac{3}{2} \frac{\epsilon}{k} P - \frac{3}{2} (1 + \beta) \frac{\epsilon^2}{k} \tag{2.8}$$

with a rather good value for $C_{\epsilon 1}$ and a value for $C_{\epsilon 2}$ which depends on the choice of β. This result is essentially Eq. (27) of [3].

Following Reynolds [8], the constant β can be fixed by appealing to the behavior of the large scales of motion during decay. Differentiation of Eq. (2.3) gives

$$\frac{\dot{k}_0}{k_0} = 3 \frac{\dot{\kappa}_0}{\kappa_0} \tag{2.9}$$

and differentiation of Eq. (2.2) gives

$$\frac{\dot{k}}{k} = \frac{2}{3} \frac{\dot{\epsilon}}{\epsilon} - \frac{2}{3} \frac{\dot{\kappa}_0}{\kappa_0} \tag{2.10}$$

Assuming that decay is self-similar, so that

$$\frac{\dot{k}}{k} = \frac{\dot{k}_0}{k_0} \tag{2.11}$$

Eqs. (2.9)–(2.11) lead as usual to

$$\frac{\dot{\epsilon}}{\epsilon} = -\frac{11}{6} \frac{\epsilon}{k} \tag{2.12}$$

corresponding, in Eq. (2.8), to $\beta = 2/9$.

It would seem that this argument solves the problem of deriving the homogeneous ϵ transport equation, since it gives the values $C_{\epsilon 1} = 3/2$ and $C_{\epsilon 2} = 11/6$. But one can object that the assumption Eq. (2.6) is another way of stating the final result: this equation states that the integral scale κ_0^{-1} satisfies a transport equation in which the production term is absent. Indeed, writing

$$\frac{d}{dt}\frac{k^{3/2}}{\epsilon} = \frac{3}{2}\frac{k^{1/2}}{\epsilon}\dot{k} - \frac{k^{3/2}}{\epsilon^2}\dot{\epsilon} \tag{2.13}$$

and substituting

$$\begin{aligned} \dot{k} &= P - \epsilon \\ \dot{\epsilon} &= \frac{\epsilon}{k}[C_{\epsilon 1}P - C_{\epsilon 2}\epsilon] \end{aligned} \tag{2.14}$$

leads to

$$\frac{d}{dt}\frac{k^{3/2}}{\epsilon} = (\frac{3}{2} - C_{\epsilon 1})\frac{k^{1/2}}{\epsilon}P + (C_{\epsilon 2} - \frac{3}{2})k^{1/2} \tag{2.15}$$

which shows that the absence of a production term in the length-scale transport equation is equivalent to $C_{\epsilon 1} = 3/2$.

The injection of energy at large scales can certainly cause the integral scale to increase; at the same time, turbulence production might be expected to it to decrease through the enhancement of small scales. Eq. (2.5) states the dominance of the first process over the second. Although the validity of this approximation is uncertain, the success of the argument is undeniable, and it seems reasonable to ask what conclusions will result if the same argument is applied to another problem.

3. Rotating turbulence. To derive an ϵ equation for rotating turbulence, we will combine the arguments of the previous section with Zhou's phenomenological model of rotating turbulence [4]. Briefly, this model postulates that strong rotation replaces the nonlinear time scale k/ϵ by the inverse rotation rate Ω^{-1}; closure theories lead to

$$\epsilon \sim \kappa^4 T E(\kappa)^2 \tag{3.1}$$

where by hypothesis, $T \propto \Omega^{-1}$, hence

$$E(\kappa) = C_K^{\Omega}\sqrt{\epsilon\Omega}\kappa^{-2} \tag{3.2}$$

For notational simplicity, Ω will denote twice the absolute value of the rotation rate throughout.

Adding a model for the large scales, we obtain the analog of the split-spectrum model of Eq. (2.1) for rotating turbulence,

$$E(\kappa) = \begin{cases} C\kappa^2 & \text{if } \kappa < \kappa_0 \\ C_K^{\Omega}\sqrt{\Omega\epsilon}\kappa^{-2} & \text{if } \kappa > \kappa_0 \end{cases} \tag{3.3}$$

Again, we have the energy of the large scales,

$$k_0 = \frac{1}{3}C\kappa_0^3 \tag{3.4}$$

and the inertial range energy

$$k = C_K^{\Omega}\sqrt{\Omega\epsilon}\kappa_0^{-1} \tag{3.5}$$

Note that the definition of the integral scale implied by Eq. (3.5) differs from the non-rotating result Eq. (2.2).

Following Schiestel, we differentiate Eq. (3.5) to obtain

$$\frac{\dot{k}}{k} = \frac{1}{2}\frac{\dot{\epsilon}}{\epsilon} - \frac{\dot{\kappa}_0}{\kappa_0} \tag{3.6}$$

As before, we must specify an equation for the inverse integral scale κ_0 in order to complete the model.

The simplest possibility is to retain Eq. (2.5). In this case, substitution of the rotation-modified spectrum Eq. (3.2) again leads to Eq. (2.6), but with a new constant of proportionality,

$$\dot{\kappa}_0/\kappa_0 = -\gamma\epsilon/k \tag{3.7}$$

Following the previous steps, we find instead of Eq. (2.8)

$$\dot{\epsilon} = 2\frac{\epsilon}{k}P - (2+2\gamma)\frac{\epsilon^2}{k} \tag{3.8}$$

with the definite prediction that $C_{\epsilon 1} = 2$ and $C_{\epsilon_2} > 2$.

The constant γ can be evaluated by assuming that the constant β in Eq. (2.5) is independent of rotation. Tentatively accepting the non-rotating result $\beta = 2/9$ suggested earlier, Eq. (2.5) with the rotation-dependent energy spectrum Eq. (3.2) leads to $\gamma = 2/9$ and to the value $C_{\epsilon 2} = 22/9$. In decaying rotating turbulence, Eq. (3.8) predicts power-law decay in time, but with a smaller exponent than non-rotating turbulence: indeed, following [8], we have

$$k(t) \sim t^{-1/(C_{\epsilon_2}-1)} \tag{3.9}$$

and the increase in $C_{\epsilon 2}$ due to rotation from 11/6 to 22/9 implies a reduction in the decay rate.

The model of rotating decaying turbulence implied by Eq. (3.8) has been advocated, for example in [5], and more recently in [9]. The value $C_{\epsilon 2} = 22/9$ in rotating turbulence can be compared to the values $C_{\epsilon 2} \approx 2.8$ recommended in [5] and $C_{\epsilon 2} = 8/3$ suggested in [9].

However, the available data is also consistent with the conclusion that in rotating turbulence, energy transfer is suppressed completely, and energy becomes trapped in the largest scales of motion, where it undergoes purely viscous decay. This picture, which is inconsistent with any kind of power-law decay, is advocated for example by [10] and [11]. Which description of decaying rotating turbulence is correct remains controversial; for now, we would like to explore some models which are consistent with the second viewpoint.

The derivation of Eq. (3.2) assumes that the time-scale in strongly rotating turbulence is the inverse rotation rate. This idea suggests replacing Eq. (2.6) by

$$\frac{\dot{\kappa}_0}{\kappa_0} = -\gamma'\Omega \tag{3.10}$$

in the strong rotation limit. Eqs. (3.6) and (3.10) yield the ϵ equation in the form

$$\dot{\epsilon} = 2\frac{\epsilon}{k}(P - \epsilon) - \gamma'\Omega\epsilon \tag{3.11}$$

The rotation dependence found in Eq. (3.11) coincides with that of the well-known Bardina model [6]; we argued previously [1] for the strong rotation limit of this model on the basis of simplified closure arguments. Integration of the Bardina model for decaying turbulence in the strong rotation limit gives the results that ϵ decays exponentially in time, but that the kinetic energy approaches a constant; if viscosity is included in the analysis, then the kinetic energy undergoes purely viscous decay.

Although these conclusions are consistent with numerical and experimental observations [10], the assumption Eq. (3.10) underlying the present derivation has the consequence that the integral scale grows exponentially. This was cited in [9] as evidence against the Bardina model itself, although [11] argued that quite different two-point behavior can be consistent with the same single-point model.

The difficulty is not so much with Schiestel's formalism, but with applying Eq. (3.10), an isotropic result, to rotating turbulence. In rotating turbulence, the Taylor-Proudman theorem forces the large scales of motion to be nearly two-dimensional. Consequently, the integral scales parallel and perpendicular to the rotation axis are unequal [10].

It is rather difficult to capture this effect in any isotropic model. But suppose that we combine Eqs. (3.6) and (2.9) to give

$$\frac{\dot{k}}{k} = \frac{1}{2}\frac{\dot{\epsilon}}{\epsilon} - \frac{1}{3}\frac{\dot{k}_0}{k_0} \tag{3.12}$$

and simply postulate the large rotation limit of Eq. (3.11) for decaying turbulence

$$\frac{\dot{\epsilon}}{\epsilon} = -\gamma'\Omega \tag{3.13}$$

Then we obtain

$$\frac{\dot{k}}{k} = -\frac{1}{2}\gamma'\Omega - \frac{1}{3}\frac{\dot{k}_0}{k_0} \tag{3.14}$$

or equivalently,

$$k k_0^{1/3} = k(0) k_0(0)^{1/3} e^{-\gamma'\Omega t/2} \tag{3.15}$$

instead of the self-similarity postulate Eq. (2.11) for non-rotating turbulence. Unlike the argument leading to Eq. (3.8), which like the derivation for isotropic turbulence assumes that the energy decay of the large scales and the inertial range scales is linked by self-similarity, the present derivation instead allows the dynamics of the large scales and the inertial range scales to be different.

The problem of decaying rotating turbulence is defined by the energy equation together with Eq. (3.13) and either Eq. (3.14) or Eq. (3.15). Numerical integration will be required to solve these equations in general, but it is evident that these equations are consistent with the limits

$$\begin{aligned} \epsilon &= 0 \\ k &= 0 \end{aligned} \tag{3.16}$$

while

$$\begin{aligned} k_0 &= \text{const.} \\ \kappa_0 &= \text{const.} \end{aligned} \tag{3.17}$$

Thus, the kinetic energy in the inertial range vanishes, the energy transfer vanishes, but the kinetic energy in the large scales and the integral scale both approach constants in the absence of viscosity.

Let us summarize the differences between the two dynamic descriptions of rotating decay. Power-law decay, but with a reduced exponent, follows if the decay of both the large-scale energy and the inertial range energy is linked through the self-similarity assumption Eq. (2.11). The alternative description, which leads instead to Eqs. (3.16) and (3.17) allows the large-scale and inertial range energies to evolve independently. The argument also implies that in the long-time limit, viscous dissipation and energy transfer are unequal: energy transfer can vanish, but viscous dissipation is always nonzero.

4. Conclusions. Schiestel's derivation of the ϵ transport equation has been generalized to rotating turbulence. By assuming that the basic scale relationship Eq. (2.5) applies to both non-rotating and rotating turbulence, we are led to the ϵ equation in the form Eq. (3.8). This equation implies algebraic decay in time of decaying rotating turbulence with a smaller decay rate than non-rotating turbulence. Replacing Eq. (3.8) with the rotation-dependent hypothesis Eq. (3.10) leads essentially to the Bardina model, which implies a completely different description of rotating decay: the nonlinear energy transfer vanishes and in the absence of viscous effects, energy approaches a constant. By ignoring the two-dimensionality and rotation-independence of the large scales, this argument leads to an incorrect description of the integral scale in decaying rotating turbulence. By modifying Schiestel's argument, the Bardina model is shown to be consistent with saturation of the integral scale.

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